International Journal of Theoretical Physics, Vol. 29, No. 3, 1990

Hamiltonian Structures Related to a New Spectral Problem

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Received June 29, 1989

A spectral problem studied by Roy Chowdhury and Swapna Roy is discussed and the corresponding Hamiltonian structure is obtained.

In a recent publication, Roy Chowdhury and Swapna Roy (1986) discussed two spectral problems. They wanted to get the Hamiltonian structures for these two hierarchies, but they did not accomplish this, because what they called the Hamiltonian operators are not even skew-symmetric. What they did was to obtain the evolution equations related to the spectral problems by the well-known AKNS procedure and put them into new forms by means of Tu's method. The designation of "Hamiltonian structures" seems inappropriate for their discussion.

Two spectral problems were discussed by Roy Chowdhury and Swapna Roy (1986). It is worthwhile to point out that the first problem, i.e.,

$$y_x = \begin{bmatrix} -i\lambda + r & p \\ q & i\lambda - r \end{bmatrix} y$$

was studied by Giachetti and Johnson (1984) and more details can be found in a recent paper by Tu (1988). In this paper, I consider the second problem.

One word on the presentation. I do not start from the basic concepts, such as Hamiltonian operators, etc. For a review of these concepts see Olver (1986) or Antonowicz and Fordy (1990).

The spectral problem under consideration reads

$$\phi_x = \begin{bmatrix} -i\lambda + r & \lambda q \\ p & i\lambda - r \end{bmatrix} \phi \tag{1}$$

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For convenience I consider the following equivalent problem:

$$\psi_x = \begin{bmatrix} -i\lambda^2 + r & \lambda q \\ \lambda p & i\lambda^2 - r \end{bmatrix} \psi \equiv U\psi \tag{2}$$

In order to derive the isospectral flows, I first solve the equation

$$V_x = [U, V] \tag{3}$$

where

$$V = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

and

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \qquad b = \sum_{i \geq 0} b_i \lambda^{-i}, \qquad c = \sum_{i \geq 0} c_i \lambda^{-i}$$

A straightforward calculation leads to the following equations:

$$\begin{aligned} a_{m,x} &= qc_{m+1} - pb_{m+1} \\ b_{m,x} &= 2rb_m - 2qa_{m+1} - 2ib_{m+2} \\ c_{m,x} &= -2rc_m + 2pa_{m+1} + 2ic_{m+2} \end{aligned} \tag{4}$$

We can solve (4) recursively and obtain a_i, b_i, c_i; we have

$$b_0 = c_0 = 0, \qquad a_0 = \alpha\,(\text{const})$$

$$a_1 = 0, \qquad b_1 = \alpha q, \qquad c_1 = \alpha p$$

$$b_2 = c_2 = 0, \qquad a_2 = \alpha qp/2i$$

$$b_3 = \alpha(2rq + iq^2 p - q_x)/2i, \qquad c_3 = \alpha(2rp + p_x + ip^2 q)/2i, \qquad a_3 = 0$$

$$b_4 = c_4 = 0, \qquad a_4 = -\alpha[4rpq + 3ip^2q^2/2 + (qp)_x - 2pq_x]/4, \qquad b_4 = c_4 = 0$$

$$a_{2k+1} = b_{2k} = c_{2k} = 0, \qquad k = 0, 1, \ldots$$

In order to show that all a_i, b_i, c_i are polynomials of r, q, p and their x derivations, I use a simple lemma of Tu (1988).

Lemma. Let U and V be two matrix functions and let (3) be satisfied. Then $(\det V)_x = 0$.

Now, by the lemma we have that $(\det)_x = (-a^2 - bc)_x = 0$, i.e., $a^2 + bc =$ const. Therefore, we have

$$\sum_{i=1}^{n-1} (b_i c_{n-1} + a_i a_{n-i}) = -a_0 a_n, \qquad n > 0$$

From the above equation one can see that the a_n are polynomials in a_i, b_i, c_i with $i<n$. At this point one can easily see that the statement follows from relation (4) and induction.

Now, according to the zero-curvature representation,

$$U_t - P_{n,x} + [U, P_n] = 0 \tag{5}$$

where $P_n \equiv (\lambda^n V)_+ \equiv V_n + V_{n-1}\lambda + \cdots + V_0\lambda^n$.

Explicitly, (5) leads to the following system of equations:

$$r_t = a_{n,x}, \qquad q_t = b_{n-1,x} - 2rb_{n-1} + 2qa_n, \qquad p_t = c_{n-1,x} + 2rc_{n-1} - 2pa_n \tag{6}$$

$$b_{n,x} = 2rb_n, \qquad c_{n,x} = -2rc_n, \qquad a_{n,x} = qc_{n+1} - pb_{n+1} \tag{7}$$

Here we take $n=2k$, and (4) shows that (7) are identities. We see that the even flow is the consistent one. At this stage, if we were to rewrite (6) in the form of a matrix, it is easy to see that the matrix operator obtained in this way would not be skew-symmetric. Therefore, we do not do this. In order to obtain the relevant Hamiltonian operator, the relation (4) should be used.

By means of (4), the evolution equations read

$$\begin{bmatrix} r \\ q \\ p \end{bmatrix}_t = \begin{bmatrix} D/2 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{bmatrix} \begin{bmatrix} 2a_n \\ c_{n+1} \\ b_{n+1} \end{bmatrix} \equiv BP^{(n)}, \qquad n = 2k \tag{8}$$

Because B is a constant skew-symmetric operator, it is self-evident that B is a Hamiltonian operator.

Now I prove that $(2a_n, c_{n+1}, b_{n+1})^T$ is a gradient of some functional. Several approaches are available for this. The easiest and most elegant is the so-called trace-identity method developed by Tu (1986).

I take $\langle x, y\rangle = \mathrm{Tr}(xy)$, and U and V satisfies equation (3). Then, according to the result of Tu (1986), $W = \lambda^\sigma V$, which is again a solution of (3); it holds that

$$\delta/\delta u_i \langle W, \partial U/\partial\lambda\rangle = \partial/\partial\lambda \langle W, \partial U/\partial u_i\rangle \tag{9}$$

Substitution of $W = \lambda^\sigma V$ into (9) leads to

$$\delta/\delta u_i \langle V, \partial U/\partial\lambda\rangle = (\lambda^{-\sigma}(\partial/\partial\lambda)\lambda^\sigma)\langle V, \partial/\partial u_i\rangle \tag{10}$$

where, in the present case, $u_1 = r$, $u_2 = q$, $u_3 = p$.

Next, I calculate the quantities needed in (10),

$$\langle V, \partial U/\partial\lambda\rangle = -4i\lambda a + bp + cq \tag{11}$$

$$\langle V, \partial U/\partial r\rangle = 2a \tag{12}$$

$$\langle V, \partial U/\partial q\rangle = \lambda c \tag{13}$$

$$\langle V, \partial U/\partial p\rangle = \lambda b \tag{14}$$

Thus, the trace identity (10) gives the following equation:

$$(\delta/\delta r, \delta/\delta q, \delta/\delta p)^T(-4ia_{n+1}+b_n p+c_n q)=(-n+1+\sigma)(2a_{n-1}, c_n, b_n) \tag{15}$$

In order to determine the value of σ, let $m=1$; it is easy to see that $\sigma=0$. Now we can rewrite equation (8) in the following Hamiltonian form:

$$\begin{pmatrix} r \\ q \\ p \end{pmatrix} = \begin{pmatrix} D/2 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} \begin{pmatrix} \delta H_n/\delta r \\ \delta H_n/\delta q \\ \delta H_n/\delta p \end{pmatrix} \tag{16}$$

where $H_n=(-4ia_{n+2}+b_{n+1}p+c_{n+1}q)/(-n)$ and we take $n=2k$.

The first nontrivial flow in this hierarchy (16) reads as follows:

$$\begin{aligned} r_t &= -i\alpha(qp)_x \\ q_t &= \alpha(-2qr-iq^2p+q_x) \\ p_t &= \alpha(-2pr+ip^2q-p_x) \end{aligned}$$

I conclude this paper with a few remarks.

Remark (i). In general, the matrix spectral problem $\phi_x=U\phi$ relates to bi-Hamiltonian structures. As a matter of fact, Fordy (1990) calculated three Hamiltonian operators for the case $U=\lambda^2\sigma_3+\lambda P+Q$, where

$$\sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P=\begin{pmatrix} u & v \\ w & -u \end{pmatrix}, \quad Q=\begin{pmatrix} s & g \\ f & -s \end{pmatrix}$$

But in the present reduction, only the second Hamiltonian operator survives.

Remark (ii). Roy Chowdhury and Swapna Roy (1986) pointed out that a coupled equation proposed by Novikov (1981) takes (1) as the spectral problem. In fact, by a simple scaling, that equation is equivalent to the mathematical disperse water wave equation. This observation suggests that the equation is tri-Hamiltonian. It appears that (1) should be related to tri-Hamiltonian structures. In general this is not the case and it might be true for some special equations in (16).

ACKNOWLEDGMENTS

I thank Prof. Gui-zhang Tu for sending me his papers prior to publication. I am also very grateful to A. P. Fordy for introducing me to this exciting field of integrable Hamiltonian systems and for valuable discussions about this subject. This research is based on my doctoral work under the Sino-British Friendship and Scholarship Scheme (SBFSS) and their financial support is gratefully acknowledged.

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